On the stability bounds in a problem of convection with uniform internal heat source

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Abstract

Two Galerkin methods are applied to a problem of convection with uniform internal heat source are given. With each method analytical results are obtained and discussed. They concern the parameter representing the heating rate. Numerical results are also given and they agree well with the existing ones.

1 The physical problem

Natural convection induced by an internal heat source is a phenomenon which has been intensively studied, especially in order to point out its influence on other processes. The motion in the atmosphere or mantle convection are two among such phenomena [15]. They bifurcate from the conduction state as a result of its loss of stability. A major importance is given to thermal convection processes in terrestrial bodies driven by internal heat sources in which the heat source is a function of time and, moreover, can vary from one terrestrial body to another. In spite of their importance, due to the occurrence of variable coefficients in the nonlinear partial differential equations governing the evolution of the perturbations around the basic equilibrium, so far these phenomena were treated mostly numerically and experimentally. In [4] we carried out a linear study for the eigenvalue problem associated with the equations for a convection problem with an uniform internal heat

source in a horizontal fluid layer bounded by two rigid walls [15]. Our method was based on Fourier series expansions for the unknown functions. Numerical results and graphs were given showing a destabilizing effect of the presence of the heat source. In [5] another two methods based on Fourier series expansions (a Chandrasekhar functions - based method and a shifted Legendre polynomials - based method) were used to study analytically the eigenvalue problem deduced in [4].

In [13] a linear stability analysis for a natural convection problem induced by internal heating is performed in order to point out the effects of the heat distribution. This is a function of both the critical Rayleigh number and the critical wavenumber. Some non-uniform distributions were considered along with the uniform one. It was shown that a concentration of the heat source near the bottom boundary implies a decreasing of the stability domain; namely it lowers the temperature difference at which the convection sets in. The variation of the critical wavenumber is small and there is only a slight influence of this distribution on the size of the convection cells. When the heat source is placed near the top boundary an enlargement of the domain of stability occurs.

Another analytical study for a problem of convection in a fluid saturated porous layer heated internally and in the presence of a linearly varying gravity field is presented in [8]. It was proved that the principle of exchange of stabilities holds as long as the gravity field and the integral of the heat source have the same sign. Convection in a medium with internal heat source was also analyzed in [1] by linear stability methods and nonlinear stability (energy type) methods. Numerical bounds for the critical value of the control parameter, the Rayleigh number, were given and the continuous dependence of the solution of the initial boundary value problem on the internal heat source was proved.

In [15] a horizontal layer of viscous incompressible fluid with constant viscosity and thermal conductivity coefficients is considered. The performed numerical investigation concerned the vertical distribution of the total fluxes and their individual components for small and moderate supercritical Rayleigh number in the presence of a uniform heat source. In this context, the heat and hydrostatic transfer equations are [15]

$$\eta = k\frac{\partial^2 \theta_B}{\partial z^2},\tag{1}$$

$$\frac{dp_B}{dz} = -\rho_B g,\tag{2}$$

where $\eta = const.$ is the heating rate, θ_B, p_B and ρ_B are the potential temperature, pressure and density in the basic state. In the fluid, the temperature at all point

varies at the same rate as the boundary temperature, so the problem is characterized by a constant potential temperature difference between the lower and the upper boundaries $\Delta\theta_B = \theta_{B_0} - \theta_{B_1}$. Taking into account (1) this leads to the following formula for the potential temperature distribution [15]

$$\theta_B = \theta_{B_0} - \frac{\Delta\theta_B}{h}\left(z + \frac{h}{2}\right) + \frac{\eta}{2k}\left[z^2 - \left(\frac{h^2}{2}\right)^2\right]. \tag{3}$$

In nondimensional variables the governing system of equations is

$$\begin{cases} \dfrac{d\mathbf{U}}{dt} = -\nabla p' + \Delta\mathbf{U} + Gr\theta'\mathbf{k}, \\ \mathrm{div}\mathbf{U} = 0, \\ \dfrac{d\theta'}{dt} = (1 - Nz)\mathbf{U}\mathbf{k} + Pr^{-1}\Delta\theta', \end{cases} \tag{4}$$

where $\mathbf{U} = (u, y, w)$ is the velocity, θ' and p' are the temperature and pressure deviations from the basic state [6], Gr is the Grashof number, Pr is the Prandtl number and N is a dimensionless parameter characterizing the heating (cooling) rate of the layer.

The boundaries are considered rigid and ideal heat conducting, so the boundary conditions read

$$\mathbf{U} = \theta' = 0 \text{ at } z = -\frac{1}{2} \text{ and } z = \frac{1}{2}. \tag{5}$$

In [4] in order to deduce the eigenvalue problem we considered the viscous incompressible fluid confined into a rectangular box bounded by two rigid walls: $V : 0 \leq x \leq a_1,\ 0 \leq y \leq a_2,\ -\dfrac{1}{2} \leq z \leq \dfrac{1}{2}$. We assumed that any unknown function in (4) is of the form from [7]

$$f(x, y, z) = \overline{F}(z)exp\left(i\left(2\pi m'\frac{x}{a_1} + 2\pi n'\frac{y}{a_2}\right)\right),$$

$m = \dfrac{2\pi m'}{a_1}, n = \dfrac{2\pi n'}{a_2}$, where $a_1 = \dfrac{L}{H}$, $a_2 = \dfrac{l}{H}$, L and l are the box sizes. Here $m' \geq 1$ and $n' \geq 1$ are the number of cells in the x and the y direction.

Another possibility is to assume disturbances periodic in x (period $2\pi/\alpha$) and y (period $2\pi/\beta$), with a growth rate σ, also of the form

$$f(x, y, z) = \overline{F}(z)exp\left(\sigma t + i\alpha x + i\beta y\right).$$

In this case, a subsequent investigation will concern the condition in which the principle of exchange of stabilities is valid.

In this paper, we treat only the stationary case and this implies that the principle of exchange of stabilities is valid. We complete our analytical study from [4], [5] with some remarks on the spectral methods used to solve the eigenvalue problem governing the linear stability of the basic state for the convection problem with uniform internal heat source.

The eigenvalue problem associated with the equations (4)-(5) in a horizontal fluid layer bounded by two rigid walls, governing the stability of the basic motion against normal mode perturbations, deduced by us in [4] has the form

$$\begin{cases} (D^2 - a^2)^2 W = \Theta, \\ (D^2 - a^2)\Theta = -a^2 R(1 - Nx)W \end{cases} \tag{6}$$

with the boundary conditions

$$W = DW = \Theta = 0 \text{ at } x = \pm\frac{1}{2}. \tag{7}$$

Here the Rayleigh number $R = Gr \cdot Pr$ represents the eigenvalue, while W, Θ, the amplitudes of the perturbations for the velocity and the temperature field respectively, form the corresponding eigenvector is (W, Θ).

2   On the convergence of the Galerkin method

In this section, we reveal some aspects of the convergence of the Galerkin method, one of the most used method for converting a differential operator boundary value problem to a discrete one.

There are more than one analytical possibilities to solve the system (6)-(7). However, some remarks on the convergence of the system are in order. First, let us perform a translation of variables $z = x + \frac{1}{2}$, such that the problem (6) becomes

$$\begin{cases} (D^2 - a^2)^2 W - \Theta = 0, \\ (D^2 - a^2)\Theta + a^2 R(N_1 - Nz)W = 0, \end{cases} \tag{8}$$

with $N_1 = 1 + \dfrac{N}{2}$ and the boundary conditions

$$W = DW = \Theta = 0 \text{ at } z = 0 \text{ and } 1. \tag{9}$$

The equations from (8) can be considered as a particular case of a more general eigenvalue problem with variable coefficients [3]

$$\begin{cases} (D^2 - a^2)^2 W = f(z)\Theta, \\ (D^2 - a^2)\Theta = -a^2 Rg(z)W, \end{cases} \tag{10}$$

on $0 \leq z \leq 1$.

The mathematical problem reads: *for given $f(z)$ and $g(z)$ (in our case $f(z) = 1$ and $g(z) = N_1 - Nz$) determine the minimum real positive R over all real positive a for which there exists a nonnul solution of the system (10)-(9).*

Following Kolomy [9] the convergence of the Galerkin method can be considered for the sixth-order equation $(D^2 - a^2)^3 W = -a^2 R(N_1 - Nz)W$ obtained by eliminating Θ between the two equations from (10). The following result holds:

Proposition 1. *The operator $L = (D^2 - a^2)^3$ is not symmetric in the sense of an $L^2(0,1)$ inner product on a space of functions satisfying $W = D^2 W = (D^2 - a^2)^2 W = 0$ at $z = 0, 1$.*

In order to prove Proposition 1, consider the inner product (LW, W^*) in $L^2(0,1)$ with W, W^* functions from $\mathcal{D}L$,

$$\mathcal{D}L := \{U \in L^2(0,1) | U = D^2 U = (D^2 - a^2)^2 U = 0 \text{ at } z = 0, 1\}.$$

The operator L is said to be symmetric if $(LW, W^*) = (W, LW^*)$ for any $W, W^* \in \mathcal{D}L$. In our case, by direct integration by parts it can be proven that $(LW, W^*) = (W, LW^*)$. However, W^* is not a function from $\mathcal{D}L$, namely W^* satisfies boundary conditions of the type

$$W^* = D^2 W^* = D(D^2 - a^2)W^* = 0 \text{ at } z = 0, 1, \tag{11}$$

whence Proposition 1. In [4] the quoted sixth order equation together with the boundary conditions (11) was investigated using spectral methods based on trigonometric Fourier series and good numerical results were obtained.

Consider the eigenvalue problem (6)-(7). Rescalling (6) by the factor $\dfrac{1}{\lambda}$, $\lambda = a^2 R$ the eigenvalue problem can be written in the form $Aw - \lambda Kw = 0$, with

$$A = \begin{pmatrix} (D^2 - a^2)^2 & 0 \\ 0 & (D^2 - a^2) \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ Nx - 1 & 0 \end{pmatrix}. \tag{12}$$

Here $w \in \mathcal{D}_A$, with $\mathcal{D}(A)$ the definition domain of the matricial differential operator A given by

$$\mathcal{D}_A := \left\{ w = (W, \Theta) \in \left(L^2 \left(-\frac{1}{2} \right), \frac{1}{2} \right)^2 | W = DW = \Theta = 0 \text{ at } z = -\frac{1}{2}, \frac{1}{2} \right\}.$$

The following convergence result was proved.

Theorem 1. *[3] Let λ be a parameter in the equation*

$$Aw - \lambda Kw = 0, \tag{13}$$

where A and K are linear operators, and the domain of A, D_A, is a linear manifold that is dense in a Hilbert space H with the inner product $(\cdot, \cdot)$. Let D_A be contained in the domain of K, and assume that the following conditions are fulfilled:

a) the operator A is a positive-definite, selfadjoint operator; that is $(Au, u) > 0$ and $(Au, v) - (v, Au) = 0$;

b) the operator $A^{-1}K$ can be extended to a completely continuous operator on the Hilbert space H_n, where H_n is the completion of D_A under the norm $(Au, u)^{1/2}$.

Then the Galerkin method for calculating the eigenvalues of (13) is a convergent process in H_n.

Using the definitions of the matricial differential operators, all the conditions of the theorem are satisfied so, the Galerkin method for computing the eigenvalues of (6) converges in the norm of H_0, with H_0 the Hilbert space obtained by completing $\mathcal{D}_A$ (which is a preHilbert space) to a Hilbert space.

Remark. Similarly, the convergence can be proved in the case of $L^2(0, 1)$.

3   Galerkin type spectral methods

The expansion functions used for the unknown fields encountered in various convection problems from hydrodynamic stability theory must have a basic property: they must be easy to evaluate. Trigonometric and polynomial functions have this property. A second requirement is the completeness of the sets of expansion functions. This assures that each function of the given space can be written as a linear combination of functions from the considered set (or, more likely, as a limit of such

a linear combination). The Chebyshev polynomials, the Legendre polynomials, the Hermite functions, the sine and cosine functions, satisfy this condition.

In the Galerkin approach used here the basis (trial) functions satisfy the boundary conditions. In this case, following Rama Rao [11], the simplest choice seems to be to write W and Θ as

$$W = \sum_{m=0}^{\infty} a_m h_{1m}(x), \quad \Theta = \sum_{m=0}^{\infty} b_m h_{2m}(x), \tag{14}$$

where $h_{1m}(x) = (1 - 4x^2)^{m+2}$, $h_{2m}(x) = (1 - 4x^2)^{m+1}$. With this choice, the unknown functions W and Θ satisfy the boundary conditions (7). Replacing these expressions in (6) and imposing the condition that the obtained equations are orthogonal to $h_{1n}(x)$ and $h_{2n}(x)$ respectively, $n \in \mathbb{N}$ we obtained an algebraic system in the unknown coefficients a_m and b_m. The condition that these coefficients are nonnull gives us the secular (dispersion) equation. However, an important remark is in order: the physical parameter N representing the heating (cooling) rate is missing from this equation.

Let us mention that the method is used in Ramma Rao [11] in a convective instability problem of a heat conducting micropolar fluid layer situated between two rigid boundaries. In order to investigate the critical values of the Rayleigh number at which instability sets in the most rough approximation is taken, with only one term for each expression from (14), so the approximate values of R are also crude. Nevertheless, in our case, for this approximation, in the classical case of Bénard convection, corresponding to $N = 0$, the critical value of the Rayleigh number R is $R = 1705.715$ for $a = 3.17$, which is a very good approximation compared to the well-known value from Chandrasekhar [2]. We can conclude that this approximation works with good results only in the classical case.

A mathematical explanation for the absence of the parameter N could be that the chosen set of expansion functions introduced an extraparity in the problem, leading to the loss of one of the physical parameter, in this case the heating (cooling) rate N.

In [5] we considered also a basis of some hyperbolic functions for the expansion of the unknown function W, i.e. $W = \sum_{n=1}^{\infty} W_n^1 C_n(x)$. For this choice the physical parameter N was not present in the dispersion equation. This is why, we assume that a more suitable choice is to consider the general case

$$W(x) = \sum_{n=1}^{\infty} W_n^1 C_n(x) + W_n^2 S_n(x)$$

where C_n and S_n are the Chandrasekhar sets of functions defined in [2]

$$\{C_n\}_{n\in\mathbb{N}}, \quad C_n(z) = \frac{\cosh(\lambda_n z)}{\cosh(\lambda_n/2)} - \frac{\cos(\lambda_n z)}{\cos(\lambda_n/2)},$$

$$\{S_n\}_{n\in\mathbb{N}}, \quad S_n(z) = \frac{\sinh(\mu_n z)}{\sinh(\mu_n/2)} - \frac{\sin(\mu_n z)}{\sin(\mu_n/2)},$$
(15)

with λ_n, μ_n given in [2] by explicit values for $n = 1, 2, 3, 4$ and by a recurrence relation for $n > 4$.

From $(6)_2$ we obtain the expression of the unknown function Θ,

$$\Theta(x) = -a^2 R \sum_{i=1}^{2} \Theta_i(x) + A\cosh(ax) + B\sinh(ax)$$

with A, B deduced from the boundary conditions $\Theta\left(\pm\dfrac{1}{2}\right) = 0$. The functions $\Theta_i(x)$, $i = 1, 2, ..., 4$, depending on the coefficients W_n^1 and W_n^2, have the form

$$
\begin{cases}
\Theta_1(x) = \sum_{n=1}^{\infty}\left\{\dfrac{W_n^1(Nx-1)\cosh(\lambda_n x)}{(\lambda_n^2-a^2)\cosh(\lambda_n/2)} - \dfrac{2N\lambda_n W_n^1 \sinh(\lambda_n x)}{(\lambda_n^2-a^2)^2\cosh(\lambda_n/2)}\right\}; \\[2mm]
\Theta_2(x) = \sum_{n=1}^{\infty}\left\{\dfrac{W_n^1(Nx-1)\cos(\lambda_n x)}{(\lambda_n^2+a^2)\cos(\lambda_n/2)} - \dfrac{2N\lambda_n W_n^1 \sin(\lambda_n x)}{(\lambda_n^2+a^2)^2\cos(\lambda_n/2)}\right\}; \\[2mm]
\Theta_3(x) = \sum_{n=1}^{\infty}\left\{\dfrac{W_n^2(Nx-1)\sinh(\mu_n x)}{(\mu_n^2-a^2)\sinh(\mu_n/2)} - \dfrac{2N\mu_n W_n^2 \cosh(\mu_n x)}{(\mu_n^2-a^2)^2\sinh(\mu_n/2)}\right\}; \\[2mm]
\Theta_4(x) = \sum_{n=1}^{\infty}\left\{\dfrac{W_n^2(Nx-1)\sin(\mu_n x)}{(\mu_n^2+a^2)\sin(\mu_n/2)} + \dfrac{2N\mu_n W_n^2 \cos(\mu_n x)}{(\mu_n^2+a^2)^2\sin(\mu_n/2)}\right\}.
\end{cases}
$$

Let us replace this expression in $(6)_1$. The orthogonality relation on C_m, S_m, $m \in \mathbb{N}$ imposed by the Galerkin procedure led us to an algebraic system for the unknown coefficients W_n^1 and W_n^2

$$
\begin{cases}
\sum_{n=1}^{\infty} W_n^1[(\lambda_n^4+a^4)\delta_{nm} - 2a^2 T_{nm}] - 2a^2 W_n^2 U_{nm} = \sum_{i=1}^{4} C_{\Theta_i} + \sum_{k=1}^{2} C_m^k; \\[2mm]
\sum_{n=1}^{\infty} -2a^2 V_{nm}W_n^1 + W_n^2[(\mu_n^4+a^4)\delta_{nm} - 2a^2 P_{nm}] = \sum_{i=1}^{4} S_{\Theta_i} + \sum_{k=1}^{2} S_n^k,
\end{cases}
$$
(16)

with

$$T_{nm} = (C_n'', C_m); \ U_{nm} = (S_n'', C_m); \ V_{nm} = (C_n'', S_m); \ P_{nm} = (S_n'', S_m)$$

and

$$C_m^1 = \int_{-1/2}^{1/2} \cosh(ax) C_m(x); \ C_m^2 = \int_{-1/2}^{1/2} \sinh(ax) C_m(x);$$

$$S_m^1 = \int_{-1/2}^{1/2} \cosh(ax) S_m(x); \ S_m^2 = \int_{-1/2}^{1/2} \sinh(ax) S_m(x);$$

$$C_{\Theta_i} = \int_{-1/2}^{1/2} \Theta_i(x) C_m(x); \ S_{\Theta_i} = \int_{-1/2}^{1/2} \Theta_i(x) S_m(x).$$

This time, the secular equation depends on N and it follows from the condition that not all these coefficients vanish. Numerical values of the Rayleigh number are then obtained and displayed in Table 1 in comparison with previous results.

N	a^2	$R_a - [4]$	$R_a - here$
0	9.711	1715.079324	1708.54
1	9.711	1711.742588	1651.04
2	9.711	1701.891001	1609.12
1	10.0	1712.257687	1651.1
4	10.0	1664.341789	1560.8
4	12.0	1685.422373	1739.2
10	9.0	1482.527042	1366.02
11	9.0	1446.915467	1366.05
12	9.00	1411.401914	1354.7

Table 1. Numerical evaluations of the Rayleigh number for various values of the parameters N and a.

For the eigenvalue problem (8)-(9), in [5], in order to avoid the loss of the parameter N different sets of orthogonal functions based on polynomials, namely on shifted Legendre polynomials (SLP) on $[0, 1]$ were proposed. The method is similar to the one presented here. Instead of $\{h_{1m}(x)\}_m$ and $\{h_{2m}(x)\}_m$ from $L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$, we used the orthogonal sets from $L^2(0, 1)$,

$$\{\beta_m(z)\}_m : \ \beta_m(z) = \int_0^z \int_0^s Q_{m+1}(t) dt ds = \frac{1}{4}\left[\frac{Q_{m+3} - Q_{m+1}}{(2m+3)(2m+5)} - \frac{Q_{m+1} - Q_{m-1}}{(2m+1)(2m+3)}\right],$$

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and

$$\{\phi_m(z)\}_m : \ \phi_m(z) = \int_0^z Q_m(t)dt = \frac{Q_{m+1} - Q_{m-1}}{2(2m+1)},$$

respectively, with Q_m the classical Legendre polynomials defined on $[-1, 1]$.

In this case, the expression of the secular equation contains the physical parameter N, so good numerical evaluations of the Rayleigh number for various values of N and a were obtained.

In [6], a general Galerkin type method is proposed for the problem written in the general form (10)-(7). The unknown function Θ is written as a Fourier series [6] of the form

$$\Theta = \sum_{m=1}^{\infty} A_m \cos(p_m z) + B_m \sin(q_m z), \tag{17}$$

where $p_m = (2m - 1)\pi$, $q_m = 2m\pi$ which implies that Θ satisfies the boundary conditions (9). The expression of Θ, introduced in $(10)_1$ leads to an expression of W in the form $W = \sum_{m=1}^{\infty} A_m f_m(z) + B_m g_m(z)$ in which the boundary conditions (7) are also considered in order to find A_m and B_m. However, in our case, the function $f(z)$ is a constant one and the application of the method in this form to (6)-(7) does not lead to a correct expression of W.

4   Conclusion

In this paper a problem of convection with uniform internal heat source is investigated. We complete a previous analytical study [4], [5] with some comments on the choice of the expansion functions and their importance for the convergence of the Galerkin method. The importance of the form of the system of ordinary differential equations which describe the eigenvalue problem governing to the linear stability of the stationary solution with respect to this choice is pointed out. We present numerical results for the new introduced methods which are similar to the ones obtained before.

The main conclusion of our analytical and numerical study performed in this paper and the previous ones is that the choice of subspaces of trial functions with respect to whom the approximation problems are solved influences the form of the algebraic system and also the numerical evaluations. The good numerical results obtained for small values of the spectral parameter are justified by the accuracy of spectral methods.

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